Relativistic particles subject to the straight line scalar potential in one-dimensional world

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No artigo são discutidas as propriedades das partículas relativísticas sujeitas ao potencial escalar de forma linear no eixo inteiro em dimensões 1+1. É demonstrado que o potencial deste tipo é potencial confinante para as partículas relativísticas. As partículas são localizadas na vizinhança de uma posição (um centro de localização) determinada pelos parâmetros do potencial. O centro de localização é comum para todos os estados das partículas. Foi demonstrado que densidade de probabilidade de posição das partículas é invariante em respeito de uma certa transformação dos parâmetros do potencial. Algumas das conseqüências das propriedades encontradas são discutidas.
Palavras-chave: equção de Dirac, sistemas em dimensoes baixas, potencial confinante.

Properties of the relativistic particles in 1+1 dimensions subject to the straight line scalar potential on the entire axis x is discussed. It is demonstrated that such a potential for the relativistic particles in 1+1 is confining one. The particles are localized in a neighborhood of a definite position (a localization center) determined by the potential parameters. The localization center is common for all states of the particles. It is shown that the position probability density of the particles is invariant under a certain transformation of the potential parameters. Some consequences of the found properties are discussed.
Keywords: Dirac equation, systema in low dimensions, confining potential.

1. INTRODUCTION

Some time ago an animated discussion of the Dirac equation in 1+1 dimensions with the scalar potential of the form $V_S = g|x|$, $g > 0$ took place in the literature [1], [2], [3], [4]. The discussion was motivated by the search of the quark-antiquark bound states and evaluation of the meson energy spectrum within that model. In the cited articles the solutions were found on semi-axis $x > 0$ and $x < 0$ and then joining at the point $x = 0$. Description of quarks in hadrons by means of the Dirac equation with scalar potential was used, for example, in [5] in comparison with the bag model of hadrons. The scalar potential used in [5] has the form $V_S = gr$, where r is the distance from the fixed point, a source of interaction. Such an approach showed good results. The form of the potential used in the above cited works [1], [2], [3], [4] modifies the potential $V_S = gr$ for the of 1+1 dimensional case. Our goal is to consider the scalar potential of the form

$$V_S = ax + b\,;\ a,b \in R\,,\ a \neq 0 \tag{1}$$

on the entire axis x with arbitrary real constants a, b except $a = 0$. We will show that the Dirac particle and the scalar particle under influence of such a potential in 1+1 dimensions exhibit properties that are not observed in higher dimensions.

2. THE DIRAC PARTICLE

First we study the Dirac particle which is described by the Dirac equation. The stationary Dirac equation with scalar potential is

$$\hat{H}\psi = E\psi , \ \hat{H} = \hat{H}_0 + \beta V_S, \ \hat{H}_0 = \alpha c\hat{p} + \beta mc^2 \tag{2}$$

where ψ is a two-component wave function $\psi = (\phi, \chi)^{\mathrm{T}}$, m is the particle mass, $\hat{p}$ is the moment operator $\hat{p} = -i\hbar\partial_x$, c is the light velocity, $\hbar$ is the Planck constant, E is the particle energy, α, β are the Dirac matrices. We use the Dirac matrices in the representation $\alpha = \sigma^2, \beta = \sigma^1$ (σ^1, σ^2 are the Pauli matrices), which is referred to as Jackiw-Rebbi representation [6] by some authors [1]. Then one can write Eq. (2) as

$$\left[\sigma^2 c\hat{p} + \sigma^1(mc^2 + V_S)\right]\psi = E\psi \tag{3}$$

Expressing the lower component χ by means of ϕ in Eq. (3)

$$\chi = \frac{1}{E}\left[ic\hat{p} + (mc^2 + V_S)\right]\phi , \ \psi \tag{4}$$

for $E \neq 0$ (the case $E = 0$ is discussed later), one gets the Schrodinger-type equation for the component ϕ

$$\left[\frac{d^2}{dx^2} + (\varepsilon^2 - V_{eff})\right]\phi = 0 \tag{5}$$

where the effective potential V_{eff} and ε are

$$V_{eff} = \frac{1}{(c\hbar)^2}(mc^2 + V_S)^2 - \frac{1}{c\hbar}\frac{dV_S}{dx}, \varepsilon = \frac{E}{c\hbar} \tag{6}$$

Eq. (5) for the potential (1) takes the form

$$\left[\frac{d^2}{dx^2} + (\tilde{E} - (Q_1 x^2 + Q_2 x))\right]\phi = 0 \tag{7}$$

with

$$\tilde{E} = \varepsilon^2 + \frac{a}{c\hbar} - \frac{(b + mc^2)^2}{(c\hbar)^2}, \ Q_1 = \frac{a^2}{(c\hbar)^2}, \ Q_2 = \frac{2a(b + mc^2)}{(c\hbar)^2} \tag{8}$$

The study of the problem (7) was completed in the textbook [7] (A.I.2), where it was demonstrated that for $Q_1 > 0$ the spectrum of the problem is only discrete and the spectrum admits all real values when $Q_1 < 0$. In the case under consideration Q_1 is square of a real quantity, Eq. (8), that is Q_1 takes only positive values. Therefore the spectrum of the problem (7) with coefficients (8) is only discrete and is given by [7]

$$\tilde{E} = \frac{|a|}{c\hbar}(2n+1) - \frac{(b + mc^2)^2}{(c\hbar)^2}, \ n = 0,1,2,..., \tag{9}$$

solutions of Eq. (7) read

$$\phi = U_n(z) = \left(2^n n!\sqrt{\pi}\right)^{-1/2} e^{-\frac{z^2}{2}} H_n(z) \tag{10}$$

where the variable z is

$$z = \left(\frac{|a|}{c\hbar}\right)^{1/2}\left(x + \frac{b + mc^2}{a}\right) \tag{11}$$

and $H_n(z)$ are Hermite polynomials.

From (9), (8) and (6) one gets the energy of the Dirac particle

$$E_n = \pm\sqrt{2c\hbar|a|n}, n = 1,2,3,... \tag{12}$$

consequently the Dirac particle admits only bound states. In Eq. (12) the sign "$\pm$" corresponds to particle and antiparticle states, respectively (the same notation is used for the eigenfunctions).

Calculating the component χ by means of Eq. (4) one gets solutions of Eq. (3), which depend on the sign of the parameter a of the potential. The solutions read

$$\psi_n^{\pm} = N \begin{pmatrix} U_n(z) \\ \pm U_{n-1}(z) \end{pmatrix}, \ a > 0; \ \psi_n^{\pm} = N \begin{pmatrix} U_{n-1}(z) \\ \mp U_n(z) \end{pmatrix}, \ a < 0; \ N = \frac{1}{\sqrt{2}} \left(\frac{|a|}{c\hbar} \right)^{1/4} \tag{13}$$

with the normalization factor N. Solutions for the zero energy state read

$$a > 0, \ \psi_0 = N_0 \begin{pmatrix} U_0(z) \\ 0 \end{pmatrix}; \ a < 0, \ \psi_0 = N_0 \begin{pmatrix} 0 \\ U_0(z) \end{pmatrix}; \ N_0 = \left(\frac{|a|}{c\hbar} \right)^{1/4} \tag{14}$$

with the normalization factor N_0.

From the fact of existence of zero energy state and from (12) one concludes that energy spectra of the particle and the antiparticle are symmetric with respect to $E = 0$.
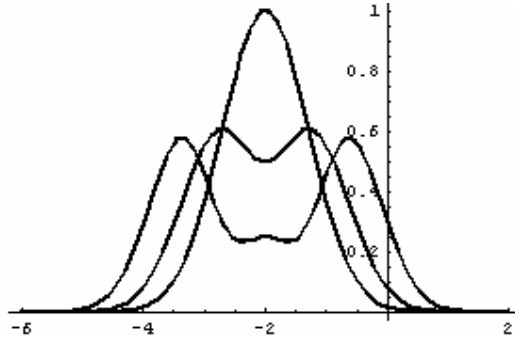

Figure 1: Position probability density $\rho(x)$ for $n = 0,1,2$, $a > 0$, $b > 0$; the corves are symmetric with respect to $\langle x \rangle = -(b + mc^2)/a$

3. PROPERTIES OF THE DIRAC PARTICLE

As the particle possesses only discrete spectrum, the particle admits only bound states and therefore is localized in the space. Calculation of the expectation value of the particle position $\langle x \rangle$ yields the value

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^{\dagger} x \psi dx = -\frac{b + mc^2}{a} \tag{15}$$

for all states of the particle including the zero energy state. The probability density of position of the particle

$$\rho(x) = \psi^{\dagger} \psi = |\phi(x)|^2 + |\chi(x)|^2 \tag{16}$$

is a symmetric function with respect to the point $\langle x \rangle$: $\rho((x - \langle x \rangle)) = \rho(-(x - \langle x \rangle))$. One can demonstrate this fact as follows. Using Eqs. (13), (14) one has

$$\rho_n(x) = \psi_n^{\dagger} \psi_n = N^2 \left([U_n(z)]^2 + [U_{n-1}(z)]^2 \right), \ n \geq 1 \tag{17}$$

$$\rho_0(x) = \psi_0^{\dagger} \psi_0 = N_0^2 [U_n(z)]^2, \ n = 0 \tag{18}$$

where

$$[U_n(z)]^2 = \left(2^n n! \sqrt{\pi} \right)^{-1} e^{-z^2} [H_n(z)]^2 \tag{19}$$

The variable z (11) can be presented as

$$z = \left(\frac{|a|}{c\hbar}\right)^{1/2} (x - \langle x \rangle) \tag{20}$$

Then one needs to show that ρ is even function of the variable z: $\rho(z) = \rho(-z)$. For $n = 0$ it is obvious. For any $n \geq 1$ one needs to show

$$[U_z(z)]^2 = [U_n(-z)]^2, \; [U_n(z)]^2 = [U_{n-1}(-z)]^2 \tag{21}$$

index of one function is even and of the other is odd. Then one has to show

$$[H_{2j}(z)]^2 = [H_{2j}(-z)]^2, \; [H_{2j+1}(z)]^2 = [H_{2j+1}(-z)]^2 \tag{22}$$

The Hermite polynomials $H_n(z)$ can be presented as

$$H_{2j}(z) = \sum_{k=0}^{l} c_{2k} z^{2k}, \; H_{2j+1}(z) = \sum_{k=0}^{l} c_{2k+1} z^{2k+1} = x \sum_{k=0}^{l} c_{2k+1} z^{2k} \tag{23}$$

with curtain coefficients c_i (see, for example, [8] (8.950.2)), then $H_{2j}(z) = H_{2j}(-z)$ and $H_{2j+1}(z) = -H_{2j}(-z)$. Therefore Eq. (22) is obeyed, consequently (21) is satisfied. Thus, the particle in every state is localized in the neighborhood of the point $\langle x \rangle$, which is determined by the particle mass and the parameters of the potential applied. We will call $\langle x \rangle$ the localization center of the particle. The curves of $\rho(x)$ for some first n's are shown on Fig. 1. (On the graphs we use the quantities in non-dimensional form

$$\overline{V}_S = \frac{V_S}{mc^2}, \; \overline{b} = \frac{b}{mc^2}, \; \overline{x} = \frac{x}{\lambda_C}, \; \overline{a} = \frac{a}{(mc^2/\lambda_C)} \tag{24}$$

where $\lambda_C = \hbar/(mc)$ is the Compton wavelength of the particle.
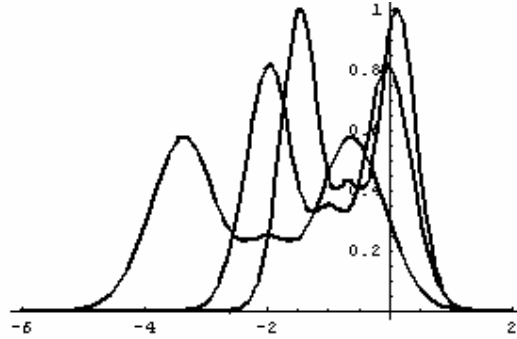

Figure 2: Position probability density $\rho(x)$ for $a = 1,2,3$; $n = 2$; $b > 0$; the curves are shifted to the right when a grows

One can also estimate the localization length of the particle l (the interval on which the particle is most probably localized). Near the localization center $\langle x \rangle$ the function $\rho(x)$ passes through a set of local extrema and goes to zero suppressed by the factor $\exp(-x^2)$ as $x \to \pm\infty$. Thus we define l as $l = 2L$, where L is determined by means of

$$\frac{\rho(\langle x \rangle + L)}{\rho(\langle x \rangle)} = e^{-1} \tag{25}$$
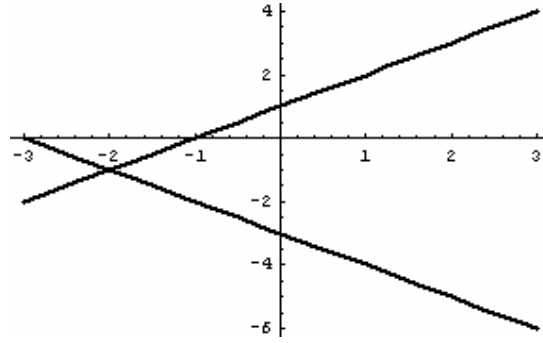
Figure 3: Potentials V_S corresponding to the same position probability density of the particle; $a = 1$, $b = 1$.

To determine l for $n \geq 1$ one has to solve the transcendental equations

$$\Lambda^2 - 1 = \ln\left\{\frac{2(2k+1)[H_{2k}(\Lambda)]^2 + [H_{2k+1}(\Lambda)]^2}{2(2k+1)[H_{2k}(0)]^2}\right\}, \quad n = 2k+1 \tag{26}$$

$$\Lambda^2 - 1 = \ln\left\{\frac{[H_{2k}(\Lambda)]^2 + 2(2k)[H_{2k-1}(\Lambda)]^2}{[H_{2k}(0)]^2}\right\}, \quad n = 2k \tag{27}$$

with respect to

$$\Lambda = \left(\frac{|a|}{c\hbar}\right)^{1/2} L \tag{28}$$

By numerical computations one finds

$$l = 2L = 2\Lambda_n \sqrt{\frac{c\hbar}{|a|}}. \tag{29}$$

where numbers $\Lambda_n > 1$ monotonically grows with n. For the zero energy state one exactly finds

$$l = 2L = 2\sqrt{\frac{c\hbar}{|a|}}. \tag{30}$$

Thus the localization length is inversely proportional to squared root of the parameter a

$$l \sim 1\big/\sqrt{|a|}. \tag{31}$$

The curves $\rho(x)$ for various values of a are presented on Fig. 2.

One can note that the distributions of ρ, Eqs. (17), (18), for and different signs of the parameter a (but for b and $|a|$ fixed) are shifted to each other (if $b \neq -mc^2$) due to the argument z (11). Although one can find that the form of ρ is same with respect to the transformations of the potential parameters

$$a \to -a, \; b \to -b - 2mc^2 \tag{32}$$

that exhibits that position probability density ρ is invariant under the transformations (32). In the other words, for two potentials $V_{S1} = ax + b$ and $V_{S2} = -ax - b - 2mc^2$ (shown on Fig.3) the one-dimensional Dirac particles exhibit the same behavior in the sense of spacial distribution. Moreover, the distribution of ρ is also same for the potential $V_{S3} = a|\xi| - mc^2$, $\xi = x - \langle x \rangle$ and both signs of a (see Fig. 3). Interaction by means of the scalar potential can also be treated as the problem with position-dependent mass $M(x) = mc^2 + V_S(x)$. Thus the problem considered is equivalent to the problem with $M(x) = a|\xi|$, $\xi = x - \langle x \rangle$. We also note that $M(x)$ at the symmetry center is $M(\langle x \rangle) = 0$.

4. THE SCALAR PARTICLE

The scalar particle is described by the Klein-Gordon equation. The scalar potential V_S is introduced into the Klein-Gordon equation by addition V_S to the mass term

$$\left[c^2 \hat{p}^2 + \left(mc^2 + V_S \right)^2 \right] \Phi = E\Phi, \tag{33}$$

where Φ is the scalar wave function. Substituting into Eq. (33) the form of V_S (1) one gets the Schrodinger-type equation for the function $\phi(x)$

$$\left[\frac{d^2}{dx^2} + \left(\tilde{E} - \left(Q_1 x^2 + Q_2 x \right) \right) \right] \phi = 0 \tag{34}$$

with

$$\tilde{E} = \frac{E^2}{(c\hbar)^2} - \frac{(b + mc^2)^2}{(c\hbar)^2}, \quad Q_1 = \frac{a^2}{(c\hbar)^2}, \quad Q_2 = \frac{2a(b + mc^2)}{(c\hbar)^2}. \tag{35}$$

Repeating the arguments presented in Sect. 2 (the paragraph after Eq. (8) one concludes that the spectrum of the problem (34) is only discrete ((A.I.2) [7]) and is given by

$$\tilde{E} = \frac{|a|}{c\hbar}(2n+1) - \frac{(b + mc^2)^2}{(c\hbar)^2}, \quad n = 0,1,2,... \tag{36}$$

the eigenfunctions read

$$\phi_n(x) = U_n(z) = \left(2^n n! \sqrt{\pi} \right)^{-1/2} e^{-\frac{z^2}{2}} H_n(z), \tag{37}$$

where the variable z is

$$z = \left(\frac{|a|}{c\hbar} \right)^{1/2} \left(x + \frac{b + mc^2}{a} \right) \tag{38}$$

and $H_n(z)$ are Hermite polynomials.

From (35), (36) one gets the energy of the scalar particle

$$E_n = \pm \sqrt{c\hbar |a|(2n+1)}, \quad n = 0,1,2,... \tag{39}$$

therefore the scalar particle also admits only bound states given by

$$\Phi_n(x) = N_\phi U_n(z), \quad N_\phi = \left(\frac{|a|}{c\hbar} \right)^{1/4}, \tag{40}$$

with the normalization factor N_ϕ. In Eq. (39) the sign "+" corresponds to the particle states and "−" to the antiparticle states. The ground energy state corresponds to $n = 0$. Thus for the scalar particle there exist the energy gap between the ground energy of the particle and antiparticle, the energy spectra of the particle and antiparticle are symmetric.

Repeating the analysis made for the Dirac equation one gets that the position expected value of the scalar particle

$$\langle x \rangle = \int \Phi_n^*(x) x \Phi_n(x) dx = -\frac{b + mc^2}{a} \tag{41}$$

is same for all states. The position probability density of the scalar particle

$$\rho_n(x) = \Phi_n^*(x)\Phi_n(x) \tag{42}$$

is symmetric with respect to $\langle x \rangle$ and the invariance of $\rho_n(x)$ under the transformaions (32) takes place for the scalar particle as well.

5. COMPLETNESS OF THE SETS OF THE SOLUTIONS

Here we discuss completeness of the sets of eigenfunctions of the Dirac equation (13), (14) and the Klein-Gordon equation (40).

5.1 Non-relativistic one-dimensional harmonic oscillator

First we prove completness of the set of eigenfunctions of the non-relativistic one-dimensional Schrodinger equation with the harmonic oscillator potential

$$\left[-\frac{d^2}{dx^2} + x^2 \right]\psi = \lambda\psi \tag{43}$$

The normalized solutions of (43) read

$$\psi_n(x) = U_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-\frac{x^2}{2}} H_n(x), \ \lambda_n = 2n + 1, \ n = 0,1,2,... \tag{44}$$

To prove completeness we evaluate the summation over all eigenfunctions (44)

$$\sum_{n=0}^{\infty} \psi_n^*(x)\psi_n(y) = \sum_{n=0}^{\infty} U_n^*(x)U_n(y) = e^{-(x^2+y^2)/2} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x)H_n(y). \tag{45}$$

First we write the sum (45) as two sums over even and odd subscripts of the Hermite polynomes

$$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x)H_n(y) =$$

$$\frac{1}{\sqrt{\pi}} \left\{ \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2k)!} H_{2k}(x)H_{2k}(y) + \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}(2k+1)!} H_{2k+1}(x)H_{2k+1}(y) \right\} = S_1 + S_2 \tag{46}$$

Thus Eq. (45) is represented as

$$\sum_{n=0}^{\infty} \psi_{2k}^*(x)\psi_{2k}(y) = e^{-(x^2+y^2)/2}(S_1 + S_2). \tag{47}$$

Then we use the relations ((8.972.2),(8.972.3) [8]) between Hermite polynomes and Laguerre polynomes for even and odd subscripts

$$H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{-1/2}(x^2), \ H_{2k+1}(x) = (-1)^k 2^{2k+1} k! x L_k^{1/2}(x^2) \tag{48}$$

and the relation ((8.335.1) [8]) for the Γ-function

$$\Gamma(2k) = \frac{2^{2k-1}}{\sqrt{\pi}} \Gamma(k)\Gamma\left(k+\frac{1}{2}\right) \tag{49}$$

to represent the factorials in the denominators (46) as

$$(2k+1)! = \Gamma(2(k+1)) = \frac{2^{2k+1}}{\sqrt{\pi}} \Gamma(k+1)\Gamma\left(k+\frac{1}{2}+1\right) \tag{50}$$

$$(2k)! = \Gamma(2k+1) = 2k\Gamma(2k) = 2k\frac{2^{2k+1}}{\sqrt{\pi}} \Gamma(k)\Gamma\left(k+\frac{1}{2}\right) = \frac{2^{2k}}{\sqrt{\pi}} \Gamma(k+1)\Gamma\left(k-\frac{1}{2}+1\right) \tag{51}$$

We consider the two sums in (46) separately. The first one is

$$S_1 = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2k)!} H_{2k}(x)H_{2k}(y) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}\left((-1)^k 2^{2k} k!\right)^2 L_k^{-1/2}(x^2)L_k^{-1/2}(y^2)}{2^{2k}\left[2^{2k}\Gamma(k+1)\Gamma\left(k-\frac{1}{2}+1\right)\right]}$$
$$= \sum_{k=0}^{\infty} \frac{k!L_k^{-1/2}(x^2)L_k^{-1/2}(y^2)}{\Gamma\left(k-\frac{1}{2}+1\right)} \tag{52}$$

The second sum in (46) is

$$S_2 = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}(2k+1)!} H_{2k+1}(x)H_{2k+1}(y) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}\left((-1)^k 2^{2k+1} k!\right)^2 xyL_k^{1/2}(x^2)L_k^{1/2}(y^2)}{2^{2k+1}\left[2^{2k+1}\Gamma(k+1)\Gamma\left(k+\frac{1}{2}+1\right)\right]}$$
$$= \sum_{k=0}^{\infty} \frac{k!xyL_k^{1/2}(x^2)L_k^{1/2}(y^2)}{\Gamma\left(k+\frac{1}{2}+1\right)} \tag{53}$$

Now we use the summation formula of Laguerre polynomes ((8.976.1) [8])

$$\sum_{k=0}^{\infty} k!\frac{L_k^{\alpha}(x)L_k^{\alpha}(y)z^k}{\Gamma(k+\alpha+1)} = \frac{(xyz)^{-\alpha/2}}{1-z} \exp\left(-z\frac{x+y}{1-z}\right) I_\alpha\left(2\frac{\sqrt{xyz}}{1-z}\right) \tag{54}$$

where $I_\alpha(x)$ are the modified Bessel functions. To use this formula we introduce a parameter $z < 1$ into the sums (52), (53)

$$S_1^{(z)} = \sum_{k=0}^{\infty} \frac{k!L_k^{-1/2}(x^2)L_k^{-1/2}(y^2)z^k}{\Gamma\left(k-\frac{1}{2}+1\right)} = \frac{\sqrt{xy}z^{1/4}}{1-z} \exp\left(-\frac{z(x^2+y^2)}{1-z}\right) I_{-1/2}\left(\frac{2xy\sqrt{z}}{1-z}\right) \tag{55}$$

$$S_2^{(z)} = \sum_{k=0}^{\infty} \frac{k!xyL_k^{1/2}(x^2)L_k^{1/2}(y^2)z^k}{\Gamma\left(k+\frac{1}{2}+1\right)} = \frac{\sqrt{xy}z^{-1/4}}{1-z} \exp\left(-\frac{z(x^2+y^2)}{1-z}\right) I_{1/2}\left(\frac{2xy\sqrt{z}}{1-z}\right) \tag{56}$$

and will take the limit $z \to 1$ at the end of calculations

$$S_1 = \lim_{z\to1-0} S_1^{(z)}, S_2 = \lim_{z\to1-0} S_2^{(z)}. \tag{57}$$

As

$$I_{1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\sinh(\xi), I_{-1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\cosh(\xi) \tag{58}$$

then

$$S_1^{(z)} = \sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{z(x^2+y^2)}{1-z}\right)\cosh\left(\frac{2xy\sqrt{z}}{1-z}\right), \tag{59}$$

$$S_2^{(z)} = \sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{z}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{z(x^2+y^2)}{1-z}\right)\sinh\left(\frac{2xy\sqrt{z}}{1-z}\right). \tag{60}$$

The products of the exponents in (59), (60) we represent as

$$\begin{aligned} R_1^{(z)} &= \sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{z(x^2+y^2)}{1-z}\right)\exp\left(\frac{2xy\sqrt{z}}{1-z}\right) = \\ &\sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{(x-y)^2}{1-z}\right)\exp\left((x^2+y^2)-\frac{2xy\sqrt{z}}{1+\sqrt{z}}\right) \end{aligned} \tag{61}$$

$$\begin{aligned} R_2^{(z)} &= \sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{z(x^2+y^2)}{1-z}\right)\exp\left(\frac{2xy\sqrt{z}}{1-z}\right) = \\ &\sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{1-z}}\exp\left(-\frac{(x+y)^2}{1-z}\right)\exp\left((x^2+y^2)+\frac{2xy}{1+\sqrt{z}}\right) \end{aligned} \tag{62}$$

then

$$S_1^{(z)} = \frac{1}{2}\left(R_1^{(z)} + R_2^{(z)}\right), S_2^{(z)} = \frac{1}{2\sqrt{z}}\left(R_1^{(z)} - R_2^{(z)}\right) \tag{63}$$

Then using the representation of the δ-function

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}}e^{-x^2/a} \to \delta(x) \text{ as } a \to 0, \tag{64}$$

one gets as $z \to 1$ from below

$$R_1^{(z)} \to \delta(x-y)e^{(x^2+y^2)-2xy} = \delta(x-y)e^{x^2}, \tag{65}$$

$$R_2^{(z)} \to \delta(x+y)e^{(x^2+y^2)+2xy} = \delta(x+y)e^{x^2}, \tag{66}$$

therefore

$$S_1^{(z)} = \frac{1}{2}\left(\delta(x-y)e^{x^2} + \delta(x+y)e^{x^2}\right) = S_1, S_2^{(z)} = \frac{1}{2}\left(\delta(x-y)e^{x^2} - \delta(x+y)e^{x^2}\right) = S_2. \tag{67}$$

Substituting (67) into (47) one comes to

$$\sum_{n=0}^{\infty}\psi_n^*(x)\psi_n(y) = e^{-(x^2+y^2)/2}(S_1 + S_2) = e^{-(x^2+y^2)/2}\delta(x-y)e^{x^2} \tag{68}$$

and finally

$$\sum_{n=0}^{\infty} \psi_n^*(x)\psi_n(y) = \sum_{n=0}^{\infty} U_n^*(x)U_n(y) = \delta(x-y), \tag{69}$$

which proves completeness of the set of the solutions of the harmonic oscilator.

5.2 Completness of the set of eigenfunctions of the Klein-Gordon equation

We estimate the summation over the eigenfunctions of the Klein-Gordon equation (40), (37), (38)

$$\sum_{n=0}^{\infty} \Phi_n^*(x)\Phi(y) = N_\phi^2 \sum_{n=0}^{\infty} U_n^*(z_1)U_n(z_2), \tag{70}$$

$$z_1 = N_\phi^2\left(x + \frac{b+mc^2}{2}\right), z_2 = N_\phi^2\left(y + \frac{b+mc^2}{2}\right). \tag{71}$$

Using Eq. (69) one gets

$$\sum_{n=0}^{\infty} U_n^*(z_1)U_n(z_2) = \delta(z_1 - z_2) = \delta\left(N_\phi^2(x-y)\right) = \frac{1}{N_\phi^2}\delta(x-y). \tag{72}$$

Therefore

$$\sum_{n=0}^{\infty} \Phi_n^*(x)\Phi(y) = N_\phi^2 \frac{1}{N_\phi^2}\delta(x-y) = \delta(x-y), \tag{73}$$

which proves completeness of the set of the solutions of the Klein-Gordon equation.

5.3 Completness of the set of eigenfunctions of the Dirac equation

We estimate the summation over the eigenfunctions of the Dirac equation (10), (11), (13), (14) both for $a > 0$ and $a < 0$

$$\sum_{n=0}^{\infty} \psi_n^\dagger(x)\psi_n(y) = N_0^2 U_0^*(z_1)U_0(z_2) + \frac{1}{2}N_0^2\sum_{n=1}^{\infty} U_0^*(z_1)U_0(z_2) + \frac{1}{2}N_0^2\sum_{n=0}^{\infty} U_0^*(z_1)U_0(z_2), \tag{74}$$

$$z_1 = N_0^2\left(x + \frac{b+mc^2}{a}\right), z_2 = N_0^2\left(y + \frac{b+mc^2}{a}\right). \tag{75}$$

Combining some terms in Eq. (74) one gets the relation

$$\sum_{n=0}^{\infty} \psi_n^\dagger(x)\psi_n(y) = \frac{1}{2}N_0^2 U_0^*(z_1)U_0(z_2) + N_0^2\sum_{n=1}^{\infty} U_n^*(z_1)U_n(z_2), \tag{76}$$

applying to which Eq. (69) one comes to

$$\sum_{n=0}^{\infty} \psi_n^\dagger(x)\psi_n(y) = \frac{1}{2}N_0^2 U_0^*(z_1)U_0(z_2) + N_0^2 \frac{1}{N_0^2}\delta(x-y) = \frac{1}{2}N_0^2 U_0^*(z_1)U_0(z_2) + \delta(x-y). \tag{77}$$

Thus the completeness of the set of eigenfunctions of the Dirac equation is under question.

1. BHALERAO, R. S.;, RAM, B.; Fun and frustration with quarkonium in a 1+1 dimension, *Am. J. Phys.* 69:817 (2001).
2. DE CASTRO, A. S.; *Am. J. Phys. 70:450 (2002)*; CAVALCANTI, R. M.; *Am. J. Phys.* 70:451 (2002).
3. HILLER, J. R.; Solution of the one-dimensional Dirac equation with a linear scalar potential, *Am. J. Phys.* 70:522 (2002); *quant-ph/0111011.*
4. NOGAMI, Y.; F. M. Toyama, VAN DIJK, W.; The Dirac equation with a confining potential, *Am. J. Phys.* 71:950 (2003).
5. CRITCHFIELD, C. L.; Scalar binding of quarks, *Phys. Rev. D* 12:923-925 (1975).
6. JACKIW, R.; REBBI, C.; Solutions with fermion number 1/2, *Physical Review D* 13:3398-3409 (1976).
7. BAGROV, V. G.; GITMAN, D. M.; Exact solutions of relativistic wave equations (Kluwer Academic Publishers, 1990).
8. GRADSHTEIN, I. S.; RYZHIK, I. W.; Table of Integrals, Series and Products (Academic Press, New York, 1994).