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Caminhadas aleatórias com perfil de memória κ-exponencial: uma primeira medição dos regimes difusivos

Random walks with memory profile of the κ-exponential: a first measurement of diffusive regimes

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Uma nova classe de caminhadas aleatórias, chamada Elephant Random Walks (ERW), com memória, não-Markoviana, foi proposta por Schütz e Trimper caracterizada por regimes difusivos típicos de difusão anômala. Seguindo esta linha, propomos um modelo de caminhadas aleatórias norteada pelo princípio de interação cinética (KIP), onde segundo o KIP a evolução temporal da função de distribuição de partículas idênticas sujeitas a colisões binárias nos remete a um funcional sempre crescente com o tempo, satisfazendo o enunciado da irreversibilidade da segunda lei da Termodinâmica. Derivamos uma distribuição κ-exponencial discreta para construir nosso modelo de caminhadas aleatórias. Realizamos simulações de Monte Carlo para quantificar os regimes difusivos típicos de caminhadas aleatórias de tamanho finito. Analisamos três regiões do parâmetro deformador κ. No limite $\kappa \to 0$, a distribuição da κ-exponencial recai na distribuição exponencial ordinária, onde encontramos a típica difusão Browniana. No limite máximo de anti-equilíbrio ($\kappa \to 3/2$), encontramos uma transição do regime difusivo ordinário para o regime superdifusivo. Adicionalmente, afastando-se da região de anti-equilíbrio, encontramos o comportamento difusivo do ERW. Nossos resultados são mais um caso de transporte anômalo em sistemas complexos, que estão associados à difusão de partículas para qual a variância se espalha de maneira não linear com o tempo.
Palavras-chave: κ-exponencial, difusão, caminhadas aleatórias.

A new class of random walks, Elephant Random Walks (ERW) in the memory type, non-Markovian, was proposed by Schütz and Trimper characterized by diffusive regimes typical of anomalous diffusion. Following this line, we proposed a model of random walks guided by the principle of kinetic interaction (KIP), where, according to KIP, the temporal evolution of the distribution function of identical particles subject to binary collisions leads us to an ever-increasing functional with time, satisfying the statement of the irreversibility of the second law of Thermodynamics. We derived a distribution of the exponential κ that is discrete. In the limit $\kappa \to 0$, the distribution of the κ-exponential falls into the ordinary exponential distribution, where we found the typical Brownian diffusion. In the upper limit of anti-equilibrium ($\kappa \to 3/2$), we found a transition from the ordinary diffusive regime to the super-diffusive regime. Additionally, moving away from the anti-equilibrium region, we found the ERW's diffusive behavior. Our results are yet another case of anomalous transport in complex systems, which are associated with particle diffusion for which the variance spreads non-linearly with time.
Key words: κ-exponential, diffusion, random walks.

1. INTRODUCTION

Stochastic processes are important for mathematics, allowing the development of several consequences in probability theory according to mathematical practice; for physics, they are present in several phenomena of physical interest. From a mathematical perspective, classical probability theory, stochastic differential equations, sigma-algebras, Martingales, fractional calculus and fractional numerical methods, time series, random walks and combinatorial analysis, Gaussian noise, Markov processes, Ornstein–Uhlenbeck process, Ito's stochastic integrals, Fokker–Planck equations, reaction–diffusion systems, jump processes, Levy processes [1-11].

In particular, random walk theory is used to investigate various phenomena in nature, including phenomena in economics, finance, ecology, physics, chemistry, biology, materials science, mental disorders and chaos. [12-19].

In an attempt to answer the question about how far the malaria mosquitoes would travel over time, Pearson, in 1905, published his questions in the journal Nature [20]. Therefore, originally, the genesis of random walks occurred in the observation of biological phenomena and, even today, several biological phenomena are modeled by random walks. Time-dependent random processes were generalized with differential equation models to describe the phenomenon of anomalous diffusion, such as Langevin equations, master equation, Fokker-Planck equation [21-23].

Another group of random walks has the peculiar characteristic of recording their decisions over time, called Elephant Random Walk (ERW) [24]. Other models of random walks with unlimited memory, which can recall decisions throughout its history, have been applied in the investigation of anomalous transport [25], amnesia [26, 27], Alzheimer's [28], the solution of new paradoxes [26], including the derivation of new random walk models with memory [29-34].

Several models were built according to the model described in [24]: the random walks model with Alzheimer's [18], the random walks model with Gaussian memory profile [25], the random walks model with exponential memory profile [26], the random walks model with a memory profile of the q-exponential [35] and the random walk model with binomial memory profile [33].

From the perspective of developing random walks altering memory characteristics, we sought to answer the guiding question of our work: What is the impact of distribution of exponential κ [36, 37] in the diffusive regimes of random walks of the memory class? To answer this question, we proposed a model of random walks with a memory profile of the exponential κ. The κ-statistics is relevant in several contexts such as in the description of pure plasmas constituted by electrons [38], cosmology [39], solar neutrinos [40], bremsstrahlung [41], anomalous diffraction [42], self-gravitating systems [43]. To achieve our goal, based on the physical relevance of the κ-statistics, we built our model of random walks with a memory profile of the κ-exponential, we reported our results in this work.

2. MATERIAL AND METHODS

In 2004, G.M. Schütz and S. Trimper solved the one-dimensional problem of non-Markovian random walks with memory, for example, at each instant of time the walker's decision is recorded. This model became known as Elephant Random Walks (ERW) [24].

In the ERW model, memory is formed by a set of random variables: $\sigma_{t'}$, where t' is the time equally chosen, every instant of time t, the walker's decision depends on its entire history, retrieved from an even distribution: $1/t$, t is the current time. In our model we replaced the uniform distribution: $1/t$ by the distribution κ-exponential - $f_\kappa(t)$.

Generalizations of Maxwell-Boltzmann entropy proposed based on fractal geometry, for the case where physical systems present weak chaos and another according to the principle of kinetic interaction, were introduced by Tsallis [44] and Kaniadakis [45, 46], respectively. A common point among these new entropies is the fact that such generalizations depend on some deforming parameter.

Kaniadakis proposed a new statistic, the κ-statistics, which generalizes the Maxwell-Boltzmann-Gibbs statistic through the variation of the deforming parameter: κ. The theory is guided by the principle of kinetic interaction (KIP). According to KIP, the temporal evolution of the distribution function of identical particles subjected to binary collisions leads us to an ever-increasing functional with time, satisfying the statement of irreversibility of the second law of Thermodynamics. Kaniadakis concluded that such a functional is related to a type of entropy defined as:

$$S_\kappa = -\langle ln_\kappa[g(x)]\rangle = -\int dx\, g(x) ln_\kappa[g(x)][g(x)] \quad (1)$$

where $g(x)$is the particle velocity distribution and ln_κ is the logarithm deformed by the parameter: κ. The ln_κ is a real and decreasing function $\forall x \in R$ quantified by:

$$ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} \tag{2}$$

its inverse is called κ-exponential [47]. For our problem we will use the discrete version of the κ-exponential

$$f_\kappa(y,\lambda) = A(\kappa,\lambda)exp_\kappa(-\lambda y) \tag{3}$$

where κ is the deforming parameter of the exponential function, λ is a positive parameter, $A(\kappa,\lambda)$ is a positive function, $exp_\kappa(-\lambda y) = \left(\sqrt{1+\lambda^2\kappa^2y^2} - \lambda\kappa y\right)^{\frac{1}{\kappa}}$ and $y = 0,1,2,3\ldots.,\infty$. The physical meaning of κ can be understood from particle correlations. The smallest index value is: $\kappa \to 3/2$, corresponds to the maximum value of the anti-equilibrium state, the state farthest from the classical thermal equilibrium, characterized by the highest correlations. In the limit of $\kappa \to +\infty$, the highest value of the index corresponds to the behavior of the system at the classic thermal limit [46].

Performing the following transformations: $y = (t - t')$, $A(\kappa,\lambda) = 1 - \left(\sqrt{1+\lambda^2\kappa^2} - \lambda\kappa\right)^{\frac{1}{\kappa}}$ and $exp_\kappa(-\lambda y) = \left(\sqrt{1+\lambda^2\kappa^2y^2} - \lambda\kappa y\right)^{\frac{1}{\kappa}} \to \left(\sqrt{1+\lambda^2\kappa^2(t-t')^2} - \lambda\kappa(t-t')\right)^{\frac{1}{\kappa}}$, it is certain that $f_\kappa(n,\lambda) \to f_\kappa(t',t)$, substituting in eq.(3), we obtained the final form of the probability density function

$$f_\kappa(t',t) = \left(1 - \left(\sqrt{1+\lambda^2\kappa^2} - \lambda\kappa\right)^{\frac{1}{\kappa}}\right) exp_\kappa\left(-\lambda(t-t')\right) \tag{4}$$

Taking the limit of $\kappa \to 0$, we obtained the ordinary exponential distribution described in [47].

$$\lim_{\kappa\to 0} f_\kappa\,(t',t) = \left(1 - exp(-\lambda)\right)exp\left(-\lambda(t-t')\right) \tag{5}$$

Decisions are recorded in memory at every moment: t, such property attributes the non-Markovian characteristic to random walking. The walker can perform one step to the right (+1) or one step to the left (-1), as in a one-dimensional Markovian random walk. According to the memory profile described by eq.(4), decisions are retrieved, directly impacting the propagation of the particle with position quantified by the stochastic evolution equation:

$$X_{t+1} = X_t + \sigma_{t+1} \tag{6}$$

At time: $t+1$, the variable σ_{t+1} takes on the value $+1$ when the walker walks one step to the right and -1 when the walker walks one step to the left. Memory consists of a set of random variables $\sigma_{t'}$ for time: $t' < t$. This process takes place as follows:

1. In the time: $t = 1$, the walker, initially in the position:X_0, where $\sigma_1 = +1$ ($\sigma_1 = -1$) with probability q $(1-q)$. The probability of the first step is:

$$P[\sigma_1 = \pm 1] = \frac{1}{2}[1 + (2q-1)\sigma_1] \tag{7}$$

2. in the time: $t+1$, a time t' is chosen from the set $\{1,2,3,\ldots,t\}$ randomly with probability: $w(t)$;

3. in the time: $t+1$, σ_{t+1} is chosen stochastically by the rule: $\sigma_{t+1} = +\sigma_{t'}$ ($\sigma_{t+1} = -\sigma_{t'}$) with probability p $(1-p)$;

$$P[\sigma_{t+1} = \pm\sigma_{t'}|\sigma_{t'}] = \frac{1}{2}[1 + (2p-1)\sigma_{t+1}\sigma_{t'}] \quad (8)$$

4. using rules (2) and (3) we obtained the time-conditioned probability: $t+1$

$$P\left[\sigma_{t+1} = \sigma\middle|\sigma_{1,2,\dots,t}\right] = \frac{1}{2}\sum_{j=1}^{t}\left[1 + (2p-1)\sigma\sigma_j\right] w(j) \quad (9)$$

where $\sigma = \pm 1$ is the observed value of the set: $\sigma_{1,2,\dots,t} = \{\sigma_1, \sigma_2, \dots, \sigma_t\}$, t $t \geq 1$.

Using eq.(9), we calculated the conditional displacement:

$$\left\langle \sigma_{t+1}^{(i)} = \sigma\middle|\sigma_{1,2,\dots,t}\right\rangle = \sum_{\sigma=\pm 1}\sigma\, P\left[\sigma_{t+1} = \sigma\middle|\sigma_{1,2,\dots,t}\right] \quad (10)$$

developing the eq.(10), we found:

$$\left\langle \sigma_{t+1}^{(i)} = \sigma\middle|\{\sigma_{1,2,\dots,t}\}\right\rangle = \frac{1}{2}\sum_{j=1}^{t}\left[1 + (2p-1)\sigma\sigma_j\right] w(j) \quad (11)$$

proceeding by solving eq.(11), we obtained:

$$\left\langle \sigma_{t+1}^{(i)} = \sigma\middle|\{\sigma_{1,2,\dots,t}\}\right\rangle = \frac{1}{2}\sum_{j=1}^{t}\alpha\, \sigma_j w(j) \quad (12)$$

proceeding, solving eq.(13), we found the general equation to find the first moment of the position

$$\langle x_{t+1}\rangle = \langle x_t\rangle + \left\langle \sum_{j=1}^{t}\alpha\, \sigma_j w(j)\right\rangle \quad (13)$$

where $\alpha = 2p - 1$ e $x_t = X_t - X_0$, is the displacement of the walker. If $w(t) = 1/t$ the model is the ERW. For more information, the reader can consult [24]. In our problem, random walks have a profile memory of the κ-exponential, we performed the following substitution: $w(t) = f_\kappa(t', t)$. So, followed by applying at the first moment of the position:

$$\langle x_{t+1}\rangle = \langle x_t\rangle + \left\langle \sum_{j=1}^{t}\alpha\, \sigma_j f_\kappa(j,t)\right\rangle \quad (14)$$

Eq.(14) does not have a closed analytical solution. So, we looked for a numerical solution. To achieve our goal, we used Monte Carlo simulation methods to estimate the variance.

$$Var(x_t) = \langle (x_t)^2\rangle - \langle x_t\rangle^2 \quad (15)$$

This type of random walk has a characteristic that the first step is macroscopically relevant, therefore, it has an impact on the diffusion regimes measured by the Hurst exponent [48]. We estimated the Hurst exponent using the asymptotic scaling law of the root mean square deviation of position with respect to time.

$$Var(x_t) = \langle (x_t)^2\rangle - \langle x_t\rangle^2 = At^{2H} \quad (16)$$

where A is a constant and H, the exponent of Hurst. For random walks, the first moment of the position grows more slowly than the second moment. Therefore, the following approximation is pertinently used: $Var(x_t) \approx \langle (x_t)^2 \rangle = At^{2H}$. The diffusive regimes are classified according to the Hurst exponent assuming the values $(2H < 1)$, $(H = 1)$ e $(2H > 1)$ in subdiffusive, ordinary diffusive and superdiffusive, respectively [48].

3. RESULTS AND DISCUSSION

We performed Monte Carlo simulations of random walks of the memory type with memory profile of Kaniadakis eq.(5). Our numerical experiments were carried out to 10^4 walkers and 10^5 steps. We explored these systems through the physical meaning of κ, understood from particle correlations. The lowest value of the index is: $\kappa \to 3/2$, corresponds to the value of the anti-equilibrium state, the state farthest from the classical thermal equilibrium, characterized by the highest correlations. In the limit of $\kappa \to +\infty$, the highest value of the index corresponds to the behavior of the system at the classical thermal limit. As an additional limiting case, we analyze the diffusive regimes for the case where the Kaniadakis discrete distribution becomes an ordinary discrete exponential distribution, for example: in the limit $\kappa \to 0$. We used, aiming to analyze the impact of κ on the diffusive regimes, the value of the decay constant: $\lambda = 1$.

After performing the numerical experiments, we performed typical measures of variance, estimating by it the Hurst exponent to characterize the diffusive regimes.

In Figure 1, we displayed the Hurst exponent values for several values of the deformer parameter: κ. We started with measures typical of the Hurst exponent, in its form: $2H$, from the anti-equilibrium limit with $\kappa \to 3/2$, increasing the deformation values to $\kappa = 5$, 100 and 1000, moving away from the anti-equilibrium state. Our finite size walks do not exhibit typical values of $2H$ for the classic thermal limit: $\kappa \to +\infty$.

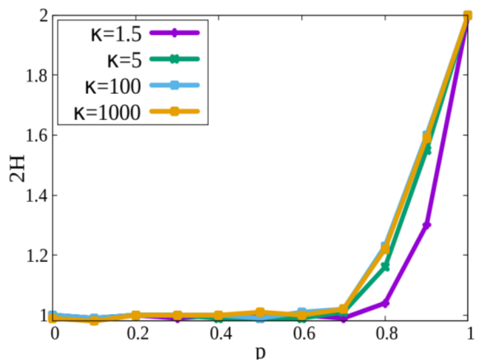

Figure 1: Behavior of the Hurst exponent (H) for different values of the feedback probability p and different values of the deforming parameter of the exponential, ranging from κ=1.5, 5.0, 100 and 1000.

In Figure 2, we displayed the Hurst exponent values for various values of the deformer parameter: κ, that is, $2H$, from the anti-equilibrium limit with $\kappa \to 3/2$, increasing the deformation values to $\kappa = 5$, 100 and 1000, moving away from the anti-equilibrium state, comparing the diffusive regimes of our model at the additional limits of the Markovian movement for $\kappa \to 0$, calculated in eq.(5) and with the diffusive regimes in the ERW model.

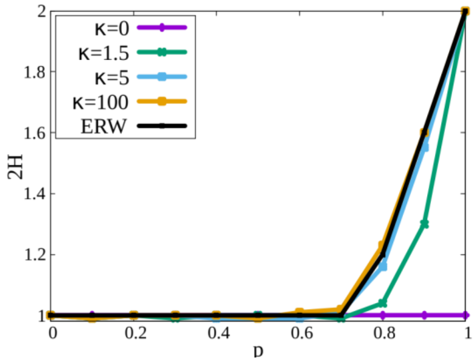

Figure 2: Behavior of the Hurst exponent (H) for several of the exponential deforming parameter: κ=0, 1.5, 5.0 and 100, comparing with the ERW model.

Let's discuss our results in ascending parameter order: κ. To $\kappa \to 0$, as expected for the exponential distribution the diffusive regimes are classic of ordinary diffusion characterized by $2H = 1$. We discussed this result outside the discussion proposed by Kaniadakis, in the limit of ordinary diffusion, to verify the robustness of our results regarding the absence of memory of the exponential function. Our discrete exponential distribution does not lead to long-range random walk correlations. However, our results do not state that random walks with an exponential memory profile cannot lead to long-range correlations. This apparent paradox is discussed in Alves et al. (2014) [26], where the authors describe a model where the exponential distribution leads to long-range correlations. Our results show us that for the discrete Kaniadakis distribution we derive, in the limit of $\kappa \to 0$, does not lead to long-range correlations characterized by $2H \neq 1$. At the limit of anti-equilibrium with $\kappa \to 3/2$, we used the value of $\kappa = 1.501$, we found a transition from the ordinary diffusive regime to the super-diffusive regime as we increased the probability p. According to our measures, the super-diffusive regime emerges first to the persistence region in $p = 0.8$ with $2H = 1.04$. Moving away from the anti-equilibrium region, we noted that the emergence of the super-diffusive regime occurs for values of p that are smaller and smaller as κ increases. This behavior occurs for the characteristic values of $\kappa = 5$, occurring in $p = 0.7$ with $2H = 1.01$ and for $\kappa = 100$, occurring in $p = 0.6$ with $2H = 1.01$. As the deformer parameter increases, the persistence region exhibits greater 2H measurements, approaching, inferiorly and superiorly, to the usual measurements of the ERW model, a behavior visually verified in Figure 2.

In this work, we proposed a model of random walks with a discrete memory profile of the exponential κ. We derived the discrete κ-exponential distribution from a generalized, discrete, exponential model used to model the propagation of defects in electronic chips. Next, we performed Monte Carlo simulations to quantify the typical diffusive regimes of finite-size random walks. We discussed our results in ascending order of the exponential deformer parameter κ. We checked that at the limit $\kappa \to 0$ the distribution of the exponential κ falls on the ordinary exponential distribution [47]. Within this limit, diffusive regimes are ordinary. We discussed this result outside the discussion proposed by Kaniadakis (2013) [45], in the limit of ordinary diffusion, to verify the robustness of our results regarding the absence of memory of the exponential function. In the upper limit of anti-equilibrium ($\kappa \to 3/2$), random walks exhibited a transition from the ordinary diffusive regime to the super-diffusive regime as we increased the probability p. We found that the super-diffusive regime initially appears in the persistence region for all values of κ. We observed that, moving away from the anti-equilibrium region, the emergence of the super-diffusive regime occurs for values of p smaller and smaller as κ increases. Additionally, we noted that as the persistence region exhibits more intense diffusive regimes, characterized by measures of $2H$ that are bigger.

4. ACKNOWLEDGMENTS

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